

The existence of the graphs that have exactly two main eigenvalues

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Abstract

An eigenvalue of a graph G is called a main eigenvalue if it has an eigenvector the sum of whose entries is not equal to zero. It is well known that a graph G has exactly two main eigenvalues if and only if there exists a unique pair of integers a and b such that $\sum_{u \in N(v)} d(u) = ad(v) + b$ for every vertex $v \in V(G)$. We collect such connected graph G in the set $\mathcal{G}(a, b)$. In this paper, we mainly focus to the existence of such a and b , and give the necessary and sufficient condition for $\mathcal{G}(a, b) \neq \emptyset$. In addition, we give the bound for the vertex degrees of $G \in \mathcal{G}(a, b)$ and use the bound to characterize the graphs in $\mathcal{G}(a, b)$ for some feasible pairs (a, b) .

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1 Introduction

Let G be a simple undirected graph with n vertices, m edges and adjacency matrix $A = A(G) = (a_{ij})_{n \times n}$. The eigenvalues of G are those of A . An eigenvalue of a graph G is called *main eigenvalue* if it has an eigenvector the sum of whose entries is not equal to zero. The Perron-Frobenius theorem implies that the largest eigenvalue of G is always main. It is well known that a graph is regular if and only if it has exactly one main eigenvalue. A long standing problem posed by Cvetković (see [1]) is to characterize graphs with exactly k ($k \geq 2$) main eigenvalues. There are some literatures on main eigenvalues and one can refer to a survey in [2]. Let \mathbf{j} denote the all-one vector. In [3], Hagos derived a simple criterion for a graph to have exactly two main eigenvalues.

Theorem [A] ([3]). A graph G has exactly two main eigenvalues if and only if there exists a unique pair of rational numbers a and b such that $A^2\mathbf{j} - aA\mathbf{j} - b\mathbf{j} = \mathbf{0}$.

$A^2\mathbf{j} - aA\mathbf{j} - b\mathbf{j} = \mathbf{0}$ can be rewritten by

$$\sum_{u \in N(v)} d(u) = ad(v) + b \quad \text{for any } v \in V(G). \quad (1)$$

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In fact, Hou and Tian in [4] showed that a and b are integers. In terms of the criterion (1), the connected graphs with $c(G) = n - m + 1 \leq 3$ that have exactly two main eigenvalues are characterized in [4, 6–11].

Let $\mathcal{G}(a, b)$ be the set of connected graphs having exactly two main eigenvalues and satisfying (1). For $G \in \mathcal{G}(a, b)$, let λ_1, λ_2 are the main eigenvalues of G . The *main polynomial* of G is defined by $m_G(x) = (x - \lambda_1)(x - \lambda_2)$. Teranishi (see [2, 5]) showed that $f(A)\mathbf{j} = \mathbf{0}$ if and only if $m_G(x)$ divides $f(x)$, where $f(x) \in \mathbb{R}[x]$. From Theorem A, we know that $A^2\mathbf{j} - a\mathbf{j} - b\mathbf{j} = \mathbf{0}$. It implies that $m_G(x) | x^2 - ax - b$ and so $m_G(x) = x^2 - ax - b$. Hence

$$\lambda_1 = \frac{a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad \lambda_2 = \frac{a - \sqrt{a^2 + 4b}}{2}. \quad (2)$$

From (2) we have $a^2 + 4b > 0$ and $a = \lambda_1 + \lambda_2 \geq 0$. Set $\mathcal{F} = \{(a, b) \mid a \geq 0 \text{ and } a^2 + 4b > 0\}$. Thus, if $\mathcal{G}(a, b) \neq \emptyset$ then $(a, b) \in \mathcal{F}$. Conversely, we ask whether $(a, b) \in \mathcal{F}$ implies $\mathcal{G}(a, b) \neq \emptyset$?

In Section 2, we give a necessary and sufficient condition for $\mathcal{G}(a, b) \neq \emptyset$. In Section 3, we obtain an upper and lower bounds for the degrees of vertices in $G \in \mathcal{G}(a, b)$ and characterize the graphs for which the lower bound is attained. In Section 4, we characterize the graphs in $\mathcal{G}(a, b)$ for some feasible pairs (a, b) .

2 The necessary and sufficient condition for $\mathcal{G}(a, b) \neq \emptyset$

Denote by $\delta(G)$, $\Delta(G)$ and $\bar{d}(G)$ the smallest, largest and mean degrees of G , respectively. First we cite useful lemma due to Rowlinson in [2].

Lemma 2.1 ([2]). *Let G be a connected graph with exactly two main eigenvalues λ_1 and λ_2 , where $\lambda_1 > \lambda_2$. Then $\lambda_2 < \delta(G) < \bar{d}(G) < \lambda_1 < \Delta(G)$.*

Now we put

$$\mathcal{F}^* = \{(a, b) \neq (0, 1) \mid a \geq 0 \text{ and } a^2 + 4b \geq 4\},$$

which we call the feasible set of the pair of integers a and b . The following result gives the necessary condition for $\mathcal{G}(a, b) \neq \emptyset$.

Lemma 2.2. *If $\mathcal{G}(a, b) \neq \emptyset$ then $(a, b) \in \mathcal{F}^*$.*

Proof. Suppose that $\mathcal{G}(a, b) \neq \emptyset$. From the above arguments in Section 1, we know that $a \geq 0$ and $a^2 + 4b > 0$, that is, $(a, b) \in \mathcal{F}$.

If $a^2 + 4b = 1$ then $a^2 = 1 - 4b$ is an odd, and so is a . From (2), we claim that $\lambda_1 = \frac{a+1}{2}$ and $\lambda_2 = \frac{a-1}{2}$ are integers. From Lemma 2.1, we have $\frac{a-1}{2} = \lambda_2 < \delta(G) < \lambda_1 = \frac{a+1}{2}$, a contradiction. If $a^2 + 4b = 2$ then $a^2 = 2(1 - 2b)$ is an even, and so is a . It follows that a^2 is a multiple of 4. This is impossible since $1 - 2b$ is an odd. If $a^2 + 4b = 3$ then $a^2 = 4(1 - b) - 1$ is an odd, and so is a . Let $a = 2k + 1$ ($k \geq 0$), we have $4k^2 + 4k + 1 = 4(1 - b) - 1$, which gives that $k^2 + k + b = \frac{1}{2}$, a contradiction. Therefore, $a^2 + 4b \geq 4$.

At last, suppose that $(a, b) = (0, 1)$. Let $v \in V(G)$ with $d(v) = \Delta(G)$, and applying (1) to v , we have $\delta(G)\Delta(G) \leq \sum_{u \in N(v)} d(u) = ad(v) + b = 1$, which implies that $G = K_2$, and so G has exactly one main eigenvalue, this is a contradiction.

Summarizing the above arguments, we known that $(a, b) \in \mathcal{F}^*$. It completes this proof. \square

For a graph G , a vertex partition $\pi: V(G) = C_1 \cup C_2 \cup \dots \cup C_r$ is said to be *equitable* if, for any $u \in C_i$, $|C_j \cap N_G(u)| = c_{ij}$ is a constant whenever $1 \leq i, j \leq r$. Such a partition π is called the *r-equitable partition* with cells C_1, C_2, \dots, C_r and parameters $(c_{11}, c_{12}, \dots, c_{1r}; c_{21}, c_{22}, \dots, c_{2r}; \dots; c_{r1}, c_{r2}, \dots, c_{rr})$. An equitable partition π leads to a *quotient graph* (or *divisor*) which is denoted by G/π . G/π is a directed graph with vertex set $\{C_1, C_2, \dots, C_r\}$, and there are exactly c_{ij} arcs from C_i to C_j . Therefore, G/π has adjacency matrix $A(G/\pi) = (c_{ij})_{r \times r}$. Rowlinson in [2] gave the following result.

Proposition 2.1 ([2]). *The main eigenvalues of G are eigenvalues of every divisor of G .*

The following result is a corollary of Proposition 2.1.

Corollary 2.1. *If G has a r -equitable partition π , then G has at most r main eigenvalues.*

Let π be a r -equitable partition of G with cells C_1, C_2, \dots, C_r and P be the character matrix whose i -th column is the character vector of C_i ($1 \leq i \leq r$). It is easy to see that $A(G)P = PA(G/\pi)$. Hence for all $\mathbf{x} \in \mathbb{R}^r$ we have

$$(\lambda I - A(G))P\mathbf{x} = P(\lambda I - A(G/\pi))\mathbf{x}.$$

Since $P\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$, it follows that \mathbf{x} is an eigenvector of $A(G/\pi)$ with eigenvalue λ if and only if $P\mathbf{x}$ is an eigenvector of $A(G)$ with eigenvalue λ . Thus, combining Proposition 2.1 we have the following result.

Lemma 2.3. *Let π be a r -equitable partition of G and P the character matrix. Then the main eigenvalues of G are the eigenvalues of $A(G/\pi)$ which have an eigenvector \mathbf{x} such that the sum of the entries of $P\mathbf{x}$ is not equal to zero.*

Let G be a connected non-regular graph having a 2-equitable partition $\pi : V(G) = C_1 \cup C_2$ with parameters $(c_{11}, c_{12}; c_{21}, c_{22})$. From Corollary 2.1 we know that G has exactly two main eigenvalues. Clearly, the induced subgraphs $G[C_1]$ is c_{11} -regular, $G[C_2]$ is c_{22} -regular, and $d_G(u) = c_{11} + c_{12}$ for $u \in C_1$, $d_G(v) = c_{21} + c_{22}$ for $v \in C_2$. Since G is non-regular, we have $c_{11} + c_{12} \neq c_{21} + c_{22}$. Applying (1) to $u \in C_1$ and $v \in C_2$, respectively, we have the following linear equations of a and b

$$\begin{cases} a(c_{11} + c_{12}) + b &= c_{11}(c_{11} + c_{12}) + c_{12}(c_{21} + c_{22}) \\ a(c_{21} + c_{22}) + b &= c_{21}(c_{11} + c_{12}) + c_{22}(c_{21} + c_{22}) \end{cases} \quad (3)$$

which gives $a = c_{11} + c_{22}$ and $b = c_{12}c_{21} - c_{11}c_{22}$, i.e., $G \in \mathcal{G}(a, b)$. Conversely, if equations (3) has the unique solution then $c_{11} + c_{12} \neq c_{21} + c_{22}$. Thus we obtain the following result.

Lemma 2.4. *Let $c_{11}, c_{22} \geq 0$, $c_{12}, c_{21} \geq 1$ be integers. Then equations (3) has the unique solution $a = c_{11} + c_{22}$ and $b = c_{12}c_{21} - c_{11}c_{22}$ if and only if there exists a graph $G \in \mathcal{G}(a, b)$ having a 2-equitable partition π with parameters $(c_{11}, c_{12}; c_{21}, c_{22})$.*

Proof. The sufficiency has established from the above arguments. For the necessity we need to construct a graph $G \in \mathcal{G}(a, b)$ having a 2-equitable partition π with parameters $(c_{11}, c_{12}; c_{21}, c_{22})$.

We first construct c_{11} -regular graph G_1 and c_{22} -regular graph G_2 such that $|V(G_1)| = |V(G_2)|$ and G_1 is connected. Without loss of generality, let $c_{11} \geq c_{22} \geq 0$. If $c_{11} = 0$ then $c_{22} = 0$. We take $G_1 = K_1$ and $G_2 = K_1$ in this case. If $c_{11} = 1$ then $c_{22} = 0$ or 1. We take $G_1 = K_2$, $G_2 = 2K_1$ or $G_2 = K_2$ in this case. If $c_{11} \geq 2$, let $t \geq c_{11} + 1$ be an even. We take $G_1 = X(\mathbb{Z}_t, S_1)$, $G_2 = tK_1$ ($c_{22} = 0$) or $G_2 = \frac{t}{2}K_2$ ($c_{22} = 1$) or $G_2 = X(\mathbb{Z}_t, S_2)$ ($c_{22} \geq 2$) in this case, where $X(\mathbb{Z}_t, S_i)$ ($i = 1, 2$) is circulant graph of order t with inverse-closed connection set S_i such that $\{1, -1\} \subseteq S_i$ and $|S_i| = c_{ii}$. Clearly, G_1 is connected since $\{1, -1\} \subseteq S_1$.

Let $V(G_1) = \{u_1, u_2, \dots, u_t\}$ and $V(G_2) = \{v_1, v_2, \dots, v_t\}$. Denote by $c_{21}G_1$ the graph consisting of c_{21} copies of G_1 and $c_{12}G_2$ the graph consisting of c_{12} copies of G_2 . Now we construct G (see Figure 1 (A)) that is obtained from $c_{21}G_1$ and $c_{12}G_2$ by adding $tc_{12}c_{21}$ edges $u_i v_i$ ($1 \leq i \leq t$) between $c_{21}G_1$ and $c_{12}G_2$. It is easy to verify that G is a connected graph with equitable partition $\pi : V(G) = C_1 \cup C_2$, where $C_1 = V(c_{21}G_1)$, $C_2 = V(c_{12}G_2)$, and the parameters of π are $(c_{11}, c_{12}; c_{21}, c_{22})$. Since equations (3) has the unique solution, we have $c_{11} + c_{12} \neq c_{21} + c_{22}$, which implies that G is non-regular, and so $G \in \mathcal{G}(a, b)$. Thus G is our required. \square

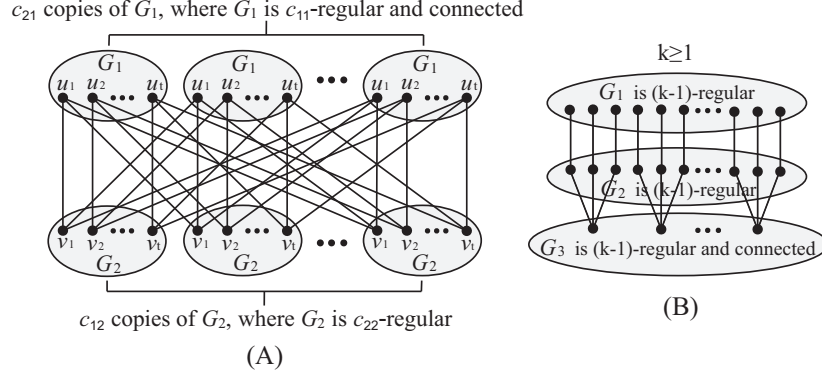


Figure 1:

Lemma 2.5. For integers a and b , if $a \geq 0$ and $a^2 + 4b > 4$ then $\mathcal{G}(a, b) \neq \emptyset$.

Proof. According to Lemma 2.4, it suffices to show that, given a and b satisfying $a \geq 0$ and $a^2 + 4b > 4$, we can find integers $c_{11}, c_{22} \geq 0$ and $c_{12}, c_{21} \geq 1$ such that $a = c_{11} + c_{22}$ and $b = c_{12}c_{21} - c_{11}c_{22}$ is the unique solution of equations (3). We will distinguish two situations in what follows.

If $a = 2k (k \geq 0)$, then $b + k^2 \geq 2$ since $a^2 + 4b > 4$. We put

$$\begin{cases} c_{11} = k, & c_{12} = b + k^2 \\ c_{21} = 1, & c_{22} = k \end{cases}$$

Then $c_{11} + c_{12} = b + k^2 + k \neq k + 1 = c_{21} + c_{22}$. Thus equations (3) has unique solution $a = c_{11} + c_{22}$, $b = c_{12}c_{21} - c_{11}c_{22}$.

If $a = 2k + 1 (k \geq 0)$, then $b + k^2 + k \geq 1$ since $a^2 + 4b > 4$. We put

$$\begin{cases} c_{11} = k + 1, & c_{12} = b + k^2 + k \\ c_{21} = 1, & c_{22} = k \end{cases}$$

Then $c_{11} + c_{12} = b + k^2 + 2k + 1 \neq k + 1 = c_{21} + c_{22}$. Thus equations (3) has unique solution $a = c_{11} + c_{22}$, $b = c_{12}c_{21} - c_{11}c_{22}$.

We complete this proof. \square

Example 2.1. According to the choices of $c_{ij} (1 \leq i, j \leq 2)$ in the proof of Lemma 2.5, for $0 \leq a \leq 5$ and b with $a^2 + 4b > 4$ we construct some classes of graphs $H_i (i = 1, 2, \dots, 6) \in \mathcal{G}(a, b)$ which are depicted in Figure 2, where $a = c_{11} + c_{22}$, $b = c_{12}c_{21} - c_{11}c_{22}$ and the corresponding parameters $c_{ij} (1 \leq i, j \leq 2)$ are shown in Table 1.

Table 1

| | k | c_{11} | c_{12} | c_{21} | c_{22} | a | $\mathcal{G}(a, b)$ |
|-------|-----|----------|----------------------|----------|----------|-----|--|
| H_1 | 0 | k | $b + k^2 \geq 2$ | 1 | k | 0 | $H_1 \in \mathcal{G}(0, b), b \geq 2$ |
| H_2 | 1 | k | $b + k^2 \geq 2$ | 1 | k | 2 | $H_2 \in \mathcal{G}(2, b), b \geq 1$ |
| H_3 | 2 | k | $b + k^2 \geq 2$ | 1 | k | 4 | $H_3 \in \mathcal{G}(4, b), b \geq -2$ |
| H_4 | 0 | $k + 1$ | $b + k^2 + k \geq 1$ | 1 | k | 1 | $H_4 \in \mathcal{G}(1, b), b \geq 1$ |
| H_5 | 1 | $k + 1$ | $b + k^2 + k \geq 1$ | 1 | k | 3 | $H_5 \in \mathcal{G}(3, b), b \geq -1$ |
| H_6 | 2 | $k + 1$ | $b + k^2 + k \geq 1$ | 1 | k | 5 | $H_6 \in \mathcal{G}(5, b), b \geq -5$ |

Remark 2.1. For some feasible $a = c_{11} + c_{22}$ and $b = c_{12}c_{21} - c_{11}c_{22}$, the choices of integers $c_{11}, c_{22} \geq 0$ and $c_{12}, c_{21} \geq 1$ are not unique. Besides the choices of Lemma 2.5, we also get $H_7 \in \mathcal{G}(2, b), H_8 \in \mathcal{G}(4, b), H_9 \in \mathcal{G}(3, b)$ (see Figure 2), and the corresponding parameters $c_{ij} (1 \leq i, j \leq 2)$ are shown in Table 2.

Table 2

| | k | c_{11} | c_{12} | c_{21} | c_{22} | a | $\mathcal{G}(a, b)$ |
|-------|-----|----------|--------------------|----------|----------|-----|--|
| H_7 | 1 | $k+1$ | $b+k^2-1 \geq 1$ | 1 | $k-1$ | 2 | $H_7 \in \mathcal{G}(2, b), \quad b \geq 1$ |
| H_8 | 2 | $k+1$ | $b+k^2-1 \geq 1$ | 1 | $k-1$ | 4 | $H_8 \in \mathcal{G}(4, b), \quad b \geq -2$ |
| H_9 | 1 | $k+2$ | $b+k^2+k-2 \geq 1$ | 1 | $k-1$ | 3 | $H_9 \in \mathcal{G}(3, b), \quad b \geq 1$ |

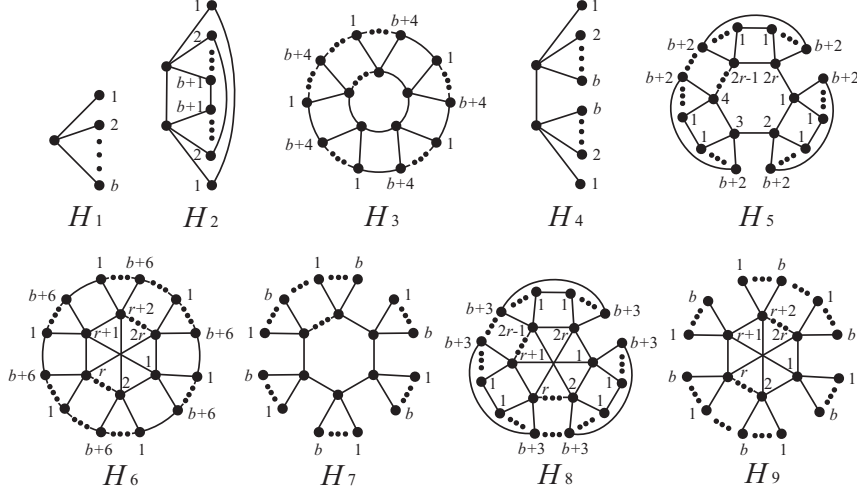


Figure 2:

Proposition 2.2. Let $G \in \mathcal{G}(a, b)$, where $a \geq 0$, $a^2 + 4b = 4$ and $(a, b) \neq (0, 1)$. Then any equitable partition of G has at least three cells.

Proof. By assumption, $a^2 = 4(1 - b)$ is an even, and so is a . Let $a = 2k$ ($k \geq 0$), we have $k \geq 1$ since $(a, b) = (0, 1)$ if $k = 0$. From Lemma 2.1 and (2) we know that $k - 1 = \lambda_2 < \delta(G) < \lambda_1 = k + 1$, and so $\delta(G) = k$.

On the contrary we may suppose that G has a 2-equitable partition $\pi : V(G) = C_1 \cup C_2$ with parameters $(c_{11}, c_{12}; c_{21}, c_{22})$. From Lemma 2.4 we know that $a = c_{11} + c_{22}$ and $b = c_{12}c_{21} - c_{11}c_{22}$, and $c_{12}, c_{21} \geq 1$ since G is connected. Without loss of generality, we assume that $d(u) = \delta(G)$ for some $u \in C_1$. Then $k = c_{11} + c_{12}$, and so $1 \leq c_{12} \leq k$. We have

$$1 \leq c_{21} = \frac{b + c_{11}c_{22}}{c_{12}} = \frac{(1 - \frac{1}{4}a^2) + c_{11}(a - c_{11})}{c_{12}} = \frac{(1 - k^2) + (k - c_{12})(k + c_{12})}{c_{12}} = \frac{1 - c_{12}^2}{c_{12}}.$$

It is impossible.

We complete this proof. \square

Lemma 2.6. For integers a and b , if $a \geq 0$, $a^2 + 4b = 4$ and $(a, b) \neq (0, 1)$, then $\mathcal{G}(a, b) \neq \emptyset$.

Proof. Note that $a \geq 0$, $a^2 + 4b = 4$ and $(a, b) \neq (0, 1)$ if and only if $a = 2k$ and $b = 1 - k^2$ for $k \geq 1$. It suffices to show $\mathcal{G}(2k, 1 - k^2) \neq \emptyset$ for $k \geq 1$. By Proposition 2.2, $G \in \mathcal{G}(2k, 1 - k^2)$ has no 2-equitable partition. In what follows we will construct $G \in \mathcal{G}(2k, 1 - k^2)$ with a 3-equitable partition $\pi : V(G) = C_1 \cup C_2 \cup C_3$.

First we can construct three $(k - 1)$ -regular graphs G_1, G_2 and G_3 with vertex sets $V(G_1) = \{u_{ij} | i = 1, 2, 3; j = 1, 2, \dots, t\}$, $V(G_2) = \{v_{ij} | i = 1, 2, 3; j = 1, 2, \dots, t\}$ and $V(G_3) = \{w_j | j = 1, 2, \dots, t\}$ such that G_3 is connected (G_3 will be K_1 if $k = 1$). Now we construct G (see Figure 1 (B)) from G_1, G_2 and G_3 by adding edges $\{u_{ij}v_{ij} | 1 \leq i \leq 3, 1 \leq j \leq t\}$ and edges $\{v_{ij}w_j | 1 \leq i \leq 3, 1 \leq j \leq t\}$. It is easy to verify that G is a connected graph with equitable

partition $\pi : V(G) = C_1 \cup C_2 \cup C_3$, where $C_1 = V(G_1)$, $C_2 = V(G_2)$, $C_3 = V(G_3)$, and the corresponding parameters c_{ij} are

$$\begin{cases} c_{11} = k - 1, & c_{12} = 1, & c_{13} = 0, \\ c_{21} = 1, & c_{22} = k - 1, & c_{23} = 1, \\ c_{31} = 0, & c_{32} = 3, & c_{33} = k - 1. \end{cases}$$

By direct calculation we know that the eigenvalues of $A(G/\pi)$ are $\lambda_1 = k + 1$, $\lambda_2 = k - 1$ and $\lambda_3 = k - 3$ with respect to eigenvectors $\mathbf{x}_1 = (1, 2, 3)^T$, $\mathbf{x}_2 = (-1, 0, 1)^T$ and $\mathbf{x}_3 = (1, -2, 3)^T$, respectively. Let P be the character matrix of π . It is easy to verify that the sums of the entries of $P\mathbf{x}_1$, $P\mathbf{x}_2$ and $P\mathbf{x}_3$ are $3t \times 1 + 3t \times 2 + t \times 3 \neq 0$, $3t \times (-1) + 3t \times 0 + t \times 1 \neq 0$ and $3t \times 1 + 3t \times (-2) + t \times 3 = 0$. From Lemma 2.3 we know that G has exactly two main eigenvalues $\lambda_1 = k + 1$ and $\lambda_2 = k - 1$, and so $G \in \mathcal{G}(a, b)$, where $a = \lambda_1 + \lambda_2 = 2k$ and $b = -\lambda_1\lambda_2 = 1 - k^2$.

We complete this proof. \square

Example 2.2. According to the choices of c_{ij} ($1 \leq i, j \leq 3$) in the proof of Lemma 2.6, for $a = 2k$ and $b = 1 - k^2$ ($k = 1, 2, 3$) we construct three graphs H_i ($i = 10, 11, 12$) $\in \mathcal{G}(a, b)$ which are depicted in Figure 3, and the corresponding parameters c_{ij} ($1 \leq i, j \leq 3$) are shown in Table 3.

Table 3

| | k | c_{11} | c_{12} | c_{13} | c_{21} | c_{22} | c_{23} | c_{31} | c_{32} | c_{33} | a | b | $\mathcal{G}(a, b)$ |
|----------|-----|----------|----------|----------|----------|----------|----------|----------|----------|----------|-----|-----|---------------------------------|
| H_{10} | 1 | $k - 1$ | 1 | 0 | 1 | $k - 1$ | 1 | 0 | 3 | $k - 1$ | 2 | 0 | $H_{10} \in \mathcal{G}(2, 0)$ |
| H_{11} | 2 | $k - 1$ | 1 | 0 | 1 | $k - 1$ | 1 | 0 | 3 | $k - 1$ | 4 | -3 | $H_{11} \in \mathcal{G}(4, -3)$ |
| H_{12} | 3 | $k - 1$ | 1 | 0 | 1 | $k - 1$ | 1 | 0 | 3 | $k - 1$ | 6 | -8 | $H_{12} \in \mathcal{G}(6, -8)$ |

Remark 2.2. The choices of c_{ij} ($1 \leq i, j \leq 3$) in the proof of Lemma 2.6 are not unique for $k \geq 2$. Besides the choices of Lemma 2.6, we also get $H_{13} \in \mathcal{G}(4, -3)$, $H_{14} \in \mathcal{G}(6, -8)$ (see Figure 3), and the corresponding parameters c_{ij} ($1 \leq i, j \leq 3$) are shown in Table 4.

Table 4

| | k | c_{11} | c_{12} | c_{13} | c_{21} | c_{22} | c_{23} | c_{31} | c_{32} | c_{33} | a | b | $\mathcal{G}(a, b)$ |
|----------|-----|----------|----------|----------|----------|----------|----------|----------|----------|----------|-----|-----|---------------------------------|
| H_{13} | 2 | $k - 1$ | 1 | 0 | 2 | $k - 2$ | 1 | 0 | 4 | $k - 1$ | 4 | -3 | $H_{13} \in \mathcal{G}(4, -3)$ |
| H_{14} | 3 | $k - 1$ | 1 | 0 | 2 | $k - 2$ | 1 | 0 | 4 | $k - 1$ | 6 | -8 | $H_{14} \in \mathcal{G}(6, -8)$ |

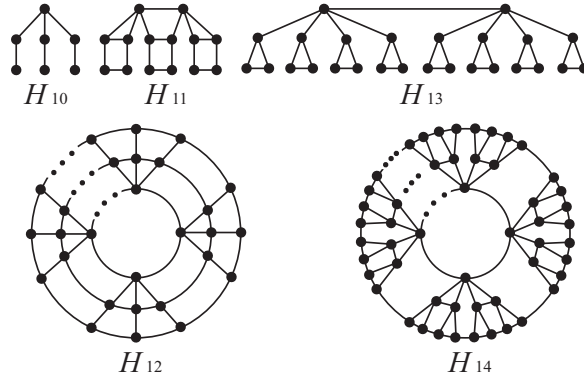


Figure 3:

From Lemmas 2.2, 2.5 and 2.6, we have the following theorem.

Theorem 2.1. $\mathcal{G}(a, b) \neq \emptyset$ if and only if $(a, b) \in \mathcal{F}^*$.

It is worth to mention that our proof of Theorem 2.1 is made of constructions. It implies that, for any pair of $(a, b) \in \mathcal{F}^*$, we can construct a graph $G \in \mathcal{G}(a, b)$ such that it has 2, or 3-equitable partition. In the next section, we will give an upper and lower bounds for the degrees of vertices in $G \in \mathcal{G}(a, b)$ and characterize graphs for which the lower bound is attained.

3 The bound for the vertex degrees of $G \in \mathcal{G}(a, b)$

In [12], Tang and Hou gave the following result.

Lemma 3.1 ([12]). *Let $G \in \mathcal{G}(a, b)$. If $a \geq 1$ then*

$$\Delta(G) \leq \frac{a^2 - a + b + 1 + \sqrt{(a^2 - a + b + 1)^2 + 4(a - 1)b}}{2} \quad (4)$$

For $G \in \mathcal{G}(a, b)$, let us define two numbers $\delta^*(G)$ and $\Delta^*(G)$ bellow:

$$\begin{aligned} \delta^*(G) &= \frac{a^2 - a\delta(G) + b + \delta(G) - \sqrt{(a^2 - a\delta(G) + b + \delta(G))^2 + 4(a - \delta(G))b}}{2}, \\ \Delta^*(G) &= \frac{a^2 - a\delta(G) + b + \delta(G) + \sqrt{(a^2 - a\delta(G) + b + \delta(G))^2 + 4(a - \delta(G))b}}{2}. \end{aligned}$$

The following result gives the lower and upper bounds for the degrees of vertices in $G \in \mathcal{G}(a, b)$, which improves the result of Lemma 3.1.

Lemma 3.2. *Let $G \in \mathcal{G}(a, b)$. For any $u \in V(G)$ we have $\delta^*(G) \leq d(u) \leq \Delta^*(G)$. Furthermore, $d(u)$ achieves its upper bound or lower bound if and only if $d(x) = \delta(G)$ for any $x \in (\cup_{v \in N(u)} N(v)) \setminus \{u\}$ (if any).*

Proof. Applying (1) to $v \in N(u)$ we have

$$ad(v) + b = \sum_{x \in N(v)} d(x) \geq d(u) + (d(v) - 1)\delta(G). \quad (5)$$

It follows that $(a - \delta(G))d(v) \geq d(u) - \delta(G) - b$. Thus

$$\begin{aligned} (a - \delta(G))(ad(u) + b) &= (a - \delta(G)) \sum_{v \in N(u)} d(v) \\ &= \sum_{v \in N(u)} (a - \delta(G))d(v) \\ &\geq d(u)(d(u) - \delta(G) - b), \end{aligned}$$

which gives

$$d(u)^2 - (a^2 - a\delta(G) + \delta(G) + b)d(u) - (a - \delta(G))b \leq 0, \quad (6)$$

$\delta^*(G) \leq d(u) \leq \Delta^*(G)$ follows immediately by solving the inequality (6).

If $d(u) = \Delta^*(G)$ (or $d(u) = \delta^*(G)$), then (6), and so (5) will be equality, which implies that $d(x) = \delta(G)$ for any $x \in (\cup_{v \in N(u)} N(v)) \setminus \{u\}$. Conversely, if $d(x) = \delta(G)$ for any $x \in (\cup_{v \in N(u)} N(v)) \setminus \{u\}$ then (5), and subsequently (6) must be equality, which implies that $d(u) = \Delta^*(G)$ or $d(u) = \delta^*(G)$.

We complete this proof. \square

From Lemma 3.2, it immediately follows the bounds for the smallest and largest degrees of $G \in \mathcal{G}(a, b)$,

$$\delta^*(G) \leq \delta(G) \text{ and } \Delta(G) \leq \Delta^*(G).$$

The first (the second) equality holds if and only if there exists a vertex $u \in V(G)$ such that $d(u) = \delta^*(G)$ ($d(u) = \Delta^*(G)$). Additionally, $\Delta(G) \leq \Delta^*(G)$ leads to (4) if $\delta(G) = 1$.

Let $\mathcal{G}_{\Delta^*}(a, b) = \{G \in \mathcal{G}(a, b) | \Delta(G) = \Delta^*(G)\}$ and $\mathcal{G}_{\delta^*}(a, b) = \{G \in \mathcal{G}(a, b) | \delta(G) = \delta^*(G)\}$. Next, we will characterize graphs in $\mathcal{G}_{\delta^*}(a, b)$.

In [2], Rowlinson gave the following result.

Lemma 3.3 ([2]). *Let G be a non-trivial connected graph with index λ . Then G is a semi-regular bipartite graph if and only if the main eigenvalues of G are λ and $-\lambda$.*

By Lemma 3.3, the following result can be obtained directly.

Theorem 3.1. *$G \in \mathcal{G}(0, b)$ if and only if G is a connected semi-regular bipartite graph with $\delta(G)\Delta(G) = b$, where $b \geq 2$.*

Theorem 3.2. $\mathcal{G}_{\delta^*}(a, b) = \mathcal{G}(0, b)$.

Proof. If $G \in \mathcal{G}(0, b)$ then $\delta(G)\Delta(G) = b$ by Theorem 3.1, which gives $\delta(G) < b$, and so $\delta^*(G) = \frac{b+\delta(G)-|b-\delta(G)|}{2} = \delta(G)$, that is $G \in \mathcal{G}_{\delta^*}(a, b)$. Conversely, if $G \in \mathcal{G}_{\delta^*}(a, b)$ then $\delta(G) = \delta^*(G)$, that is

$$\delta(G) = \frac{a^2 - a\delta(G) + b + \delta(G) - \sqrt{(a^2 - a\delta(G) + b + \delta(G))^2 + 4(a - \delta(G))b}}{2},$$

and so

$$(a^2 - a\delta(G) + b - \delta(G))^2 - (a^2 - a\delta(G) + b + \delta(G))^2 - 4(a - \delta(G))b = 0.$$

It follows that $4a(a\delta(G) + b - \delta(G)^2) = 0$, which implies that $a = 0$, and so $G \in \mathcal{G}(0, b)$. Otherwise, $a\delta(G) + b - \delta(G)^2 = 0$. Let $u \in V(G)$ and $d(u) = \delta(G)$. By applying (1) to u , we have $\sum_{v \in N(u)} d(v) = a\delta(G) + b = \delta(G)^2$, which leads to $d(v) = \delta(G)$ for any $v \in N(u)$. By regarding $v \in N(u)$ as u , and repeating this procedure, we know that G is a $\delta(G)$ -regular graph, which contradicts $G \in \mathcal{G}(a, b)$.

We complete this proof. \square

At the last of this section we propose a problem to characterize the graphs in $\mathcal{G}_{\Delta^*}(a, b)$.

4 Characterize the graphs in $\mathcal{G}(a, b)$ for some $(a, b) \in \mathcal{F}^*$

Let \mathcal{G} be the set of the connected graphs having exactly two main eigenvalues. The graphs in \mathcal{G} would be partitioned as trees, unicyclic, bicyclic and tricyclic graphs,..., and so on. The graphs in \mathcal{G} are characterized on this way by many authors and come up against tricyclic graphs (to refer [4, 6–11] for references).

Theorem 2.1 leads to a new partition: $\mathcal{G} = \cup_{(a,b) \in \mathcal{F}^*} \mathcal{G}(a, b)$. It provides us a classification of \mathcal{G} such that we can advance forward this work. $\mathcal{G}(0, b) (b \geq 2)$ is determined in Theorem 3.1. In this section we will characterize the graphs in $\mathcal{G}(2, 0)$ and $\mathcal{G}(1, b)$ for $b = 1, 2, 3, 4, 5$.

Let $a (a \geq 2)$ be an integer. Denote by T_a the rooted tree with root u , where $d(u) = a^2 - a + 1$, $d(v) = a$ for any $v \in N(u)$ and $d(x) = 1$ for any $x \in V(T_a) \setminus (\{u\} \cup N(u))$. For example, H_{10} depicted in Figure 3 is T_2 . Using (1), it is easy to verify that $T_a \in \mathcal{G}(a, 0)$.

Theorem 4.1. $\mathcal{G}(2, 0) = \{T_2\}$.

Proof. First we know that $T_2 \in \mathcal{G}(2, 0)$. Let $G \in \mathcal{G}(2, 0)$, from Lemma 2.1 and (2), we have $0 = \lambda_2 < \delta(G) < \lambda_1 = 2 < \Delta(G)$, which gives $\delta(G) = 1$ and $\Delta(G) \geq 3$. By simple calculation, we have $\Delta^*(G) = 3$, and so $\Delta(G) = 3$. Let $u \in V(G)$, $d(u) = 3$ and $N(u) = \{v_1, v_2, v_3\}$. Applying (1) to u , we have $d(v_1) + d(v_2) + d(v_3) = 6$, which implies that $d(v_1) = d(v_2) = d(v_3) = 2$. Otherwise, there exists a vertex in $N(u)$, say v_1 , with $d(v_1) = 1$. Applying (1) to v_1 , we have $d(u) = 2$, a contradiction. Let $w_i = N(v_i) \setminus \{u\} (i = 1, 2, 3)$, we have $d(w_i) = \delta(G) = 1$ by Lemma 3.2. Thus $G = T_2$. \square

Let $G \in \mathcal{G}(1, b)$, from Theorem 2.1 we know that $a^2 + 4b \geq 4$, which gives $b \geq 1$.

Lemma 4.1. *Let $G \in \mathcal{G}(1, b)$. If $b \in \{1, 2\}$ then $\delta(G) = 1$; if $b \in \{3, 4, 5, 6\}$ then $\delta(G) \leq 2$.*

Proof. From Lemma 2.1 and (2), we have $\delta(G) < \lambda_1 = \frac{1+\sqrt{1^2+4b}}{2}$. By replacing b in $\frac{1+\sqrt{1^2+4b}}{2}$ with 1, 2, 3, 4, 5 and 6, respectively, our result will be verified. \square

A double star S_{n_1, n_2} is a graph of order $n_1 + n_2 + 2$ obtained from an edge uv by adding $n_1 + n_2$ pendant edges $uu_1, uu_2, \dots, uu_{n_1}, vv_1, vv_2, \dots, vv_{n_2}$, where $n_1 \geq 1$ and $n_2 \geq 1$.

Lemma 4.2. *If $G \in \mathcal{G}(1, b)$ and $\delta(G) = 1$, then $G = S_{b, b}$.*

Proof. By simple calculation, we have $\Delta^*(G) = 1 + b$. Let $v_1 \in V(G)$ be a pendant vertex and u be the unique neighbor of v_1 . Applying (1) to v_1 , we have $d(u) = 1 + b = \Delta^*(G)$. Let $N(u) = \{v_1, v_2, \dots, v_{1+b}\}$. Applying (1) to u , we have $\sum_{i=1}^{1+b} d(v_i) = 1 + 2b$, which implies that there exists a vertex in $N(u)$, say v_{1+b} , with $d(v_{1+b}) \geq 2$. Let $w_1 \in N(v_{1+b}) \setminus \{u\}$, we have $d(w_1) = 1$ by Lemma 3.2, and again get $d(v_{1+b}) = 1 + b$ by applying (1) to w_1 . Since $\sum_{i=1}^{1+b} d(v_i) = 1 + 2b$, we have $d(v_1) = d(v_2) = \dots = d(v_b) = 1$. Let $N(v_{1+b}) = \{u, w_1, w_2, \dots, w_b\}$. Similarly, we obtain $d(w_1) = d(w_2) = \dots = d(w_b) = 1$ by regarding v_{1+b} as u in above arguments. Hence $G = S_{b, b}$. \square

Denote by $\mathcal{G}_1(1, b)$ ($b \geq 1$) the set of all the connected graphs in which every graph has a 2-equitable partition $\pi : V(G) = C_1 \cup C_2$ with parameters $(1, i; j, 0)$, where $i, j \geq 1$, $ij = b$ and $j \neq 1 + i$. Using (1), it is easy to verify that $\mathcal{G}_1(1, b) \subseteq \mathcal{G}(1, b)$.

Obviously, $S_{b, b} \in \mathcal{G}_1(1, b)$ and the corresponding parameters are $(1, b; 1, 0)$.

Theorem 4.2. $\mathcal{G}(1, 1) = \{S_{1, 1}\}$, $\mathcal{G}(1, 2) = \{S_{2, 2}\}$.

Proof. First we know that $S_{b, b} \in \mathcal{G}_1(1, b) \subseteq \mathcal{G}(1, b)$. Let $G \in \mathcal{G}(1, b)$ and $b \in \{1, 2\}$. Then $\delta(G) = 1$ by Lemma 4.1, and so $G = S_{b, b}$ by Lemma 4.2. \square

For $b \geq 3$, let $\mathcal{G}_1(1, b) = \{G \in \mathcal{G}_1(1, b) | (1, i; j, 0) = (1, 1; b, 0)\}$. Let $\mathcal{G}_2(1, 4) = \{G \in \mathcal{G}_1(1, 4) | (1, i; j, 0) = (1, 2; 2, 0)\}$. We have the following result.

Lemma 4.3. *If $G \in \mathcal{G}(1, b)$ and $\delta(G) = 2$, then $1 + \frac{b}{2} \leq \Delta(G) \leq b$. In particular, if $\Delta(G) = b$ then $G \in \mathcal{G}_1(1, b)$.*

Proof. First we claim that $b \geq 3$, for otherwise $\delta(G) = 1$ by Lemma 4.1. Let $v \in V(G)$ and $d(v) = 2$. Applying (1) to v , we have $2 + b = \sum_{u \in N(v)} d(u) \leq 2\Delta(G)$, which gives $1 + \frac{b}{2} \leq \Delta(G)$.

On the other hand, from Lemma 3.2 we have $\Delta(G) \leq \Delta^*(G) = \frac{b+1+\sqrt{(b-1)^2}}{2} = b$.

If $\Delta(G) = b$, let $u \in V(G)$, $d(u) = b$ and $N(u) = \{v_1, v_2, \dots, v_b\}$. Applying (1) to u , we have $\sum_{i=1}^b d(v_i) = 2b$, which implies that $d(v_1) = d(v_2) = \dots = d(v_b) = 2$ since $\delta(G) = 2$. Let $N(v_i) = \{u, w_i\}$ ($i = 1, 2, \dots, b$). Since $d(u) = \Delta^*(G)$, from Lemma 3.2 we have $d(w_1) = d(w_2) = \dots = d(w_b) = \delta(G) = 2$. Let $N(w_i) = \{v_i, x_i\}$ ($i = 1, 2, \dots, b$). Applying (1) to w_i , we have $2 + b = ad(w_i) + b = d(v_i) + d(x_i) = 2 + d(x_i)$, which gives $d(x_1) = d(x_2) = \dots = d(x_b) = b$. By regarding x_i as u , and repeating above procedure, we know that $d(y) = 2$ or b for any $y \in V(G)$, and that if $d(y) = b$ then $d(z) = 2$ for any $z \in N(y)$; if $d(y) = 2$ then the degrees of the two neighbors of y are 2 and b , respectively. Let

$$\begin{aligned} C_1 &= \{u \in V(G) | d(u) = 2\}, \\ C_2 &= \{u \in V(G) | d(u) = b\}. \end{aligned}$$

Then $\pi : V(G) = C_1 \cup C_2$ is a partition of $V(G)$. From the above arguments we know that $|N(u) \cap C_1| = 1$ and $|N(u) \cap C_2| = 1$ for any $u \in C_1$, $|N(u) \cap C_1| = b$ and $|N(u) \cap C_2| = 0$ for any $u \in C_2$. Therefore, $\pi : V(G) = C_1 \cup C_2$ is a 2-equitable partition of $V(G)$ with parameters $(1, 1; b, 0)$. By the definition, $G \in \mathcal{G}_1(1, b)$. \square

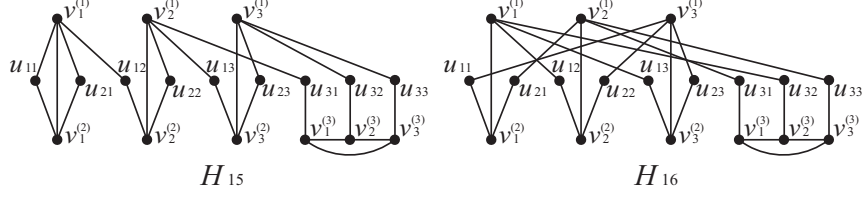


Figure 4:

Given $t \geq 3$, now we define a graph $G = (V(G), E(G))$ with a partition

$$V(G) = V_1 \cup V_2 \cup V_3 \cup V_4, \quad (7)$$

where $V_1 = \{v_1^{(1)}, v_2^{(1)}, \dots, v_t^{(1)}\}$, $V_2 = \{v_1^{(2)}, v_2^{(2)}, \dots, v_t^{(2)}\}$, $V_3 = \{v_1^{(3)}, v_2^{(3)}, \dots, v_t^{(3)}\}$ and V_4 has exactly $3t$ vertices which are decomposed into three parts of the same cardinality: $V'_{i4} = \{u'_{i1}, u'_{i2}, \dots, u'_{it}\}$ for $i = 1, 2, 3$. $G[V_i]$ is a vertex independent set for $i = 1, 2, 4$, $G[V_3]$ is a 2-regular subgraph. The edges of G between V_1 and V_4 are defined to be

$$\{v_i^{(1)} u'_{ji} \mid i = 1, 2, \dots, t; j = 1, 2, 3\}.$$

Relabeling the vertices in V_4 arbitrarily such that they are decomposed into another three parts of the same cardinality: $V_{i4} = \{u_{i1}, u_{i2}, \dots, u_{it}\}$ for $i = 1, 2, 3$. The other edges of G are defined to be

$$\begin{cases} \text{the edges between } V_1 \text{ and } V_2: \{v_i^{(1)} v_i^{(2)} \mid i = 1, \dots, t\}, \\ \text{the edges between } V_2 \text{ and } V_4: \{v_i^{(2)} u_{1i}, v_i^{(2)} u_{2i} \mid i = 1, \dots, t\}, \\ \text{the edges between } V_3 \text{ and } V_4: \{v_i^{(3)} u_{3i} \mid i = 1, \dots, t\}. \end{cases}$$

We collect such connected graph G in the set $\mathcal{G}_2(1, 5)$. For example, the graphs H_{15} and H_{16} depicted in Figure 4 are in $\mathcal{G}_2(1, 5)$.

By definition, $G \in \mathcal{G}_2(1, 5)$ has vertex partition (7), and $d(v_i)$ and $\sum_{u \sim v_i} d(u)$ are constant for any $v_i \in V_i$ (for example, any $v_1 \in V_1$ has four neighbors whose degrees are 2, 2, 2 and 3, respectively, and so $\sum_{u \sim v_1} d(u) = 3 \cdot 2 + 3 = 9$). Now we take $v_i \in V_i$ for $i = 1, 2, 3, 4$, and put

$$\begin{cases} a \cdot 4 + b = ad(v_1) + b = \sum_{u \in N(v_1)} d(u) = 3 \cdot 2 + 3 = 9, \\ a \cdot 3 + b = ad(v_2) + b = \sum_{u \in N(v_2)} d(u) = 2 \cdot 2 + 4 = 8, \\ a \cdot 3 + b = ad(v_3) + b = \sum_{u \in N(v_3)} d(u) = 2 \cdot 3 + 2 = 8, \\ a \cdot 2 + b = ad(v_4) + b = \sum_{u \in N(v_4)} d(u) = 3 + 4 = 7. \end{cases}$$

The above equation gives an unique solution $a = 1, b = 5$. It immediately follows the following result by (1).

Lemma 4.4. $\mathcal{G}_2(1, 5) \subseteq \mathcal{G}(1, 5)$.

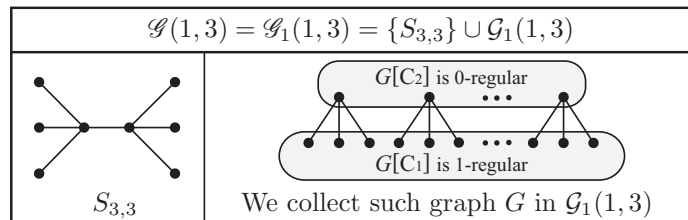


Figure 5: All the graphs in $\mathcal{G}(1, 3)$

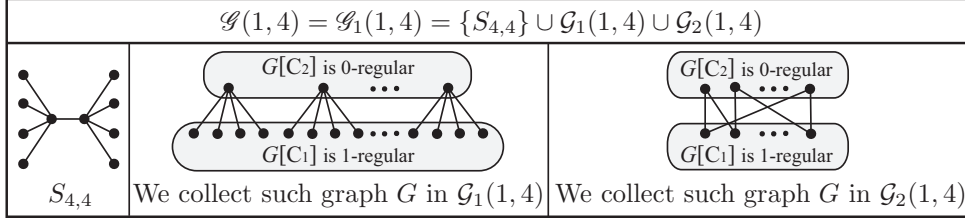


Figure 6: All the graphs in $\mathcal{G}(1, 4)$

Theorem 4.3. $\mathcal{G}(1, 3) = \mathcal{G}_1(1, 3)$, $\mathcal{G}(1, 4) = \mathcal{G}_1(1, 4)$ and $\mathcal{G}(1, 5) = \mathcal{G}_1(1, 5) \cup \mathcal{G}_2(1, 5)$.

Proof. First we know that $\mathcal{G}_1(1, b) \subseteq \mathcal{G}(1, b)$ and $\mathcal{G}_2(1, 5) \subseteq \mathcal{G}(1, 5)$. Let $G \in \mathcal{G}(1, b)$ and $b \in \{3, 4, 5\}$. Then $\delta(G) = 1$ or 2 by Lemma 4.1. Suppose that $\delta(G) = 1$. Then $G = S_{b,b} \in \mathcal{G}_1(1, b)$ by Lemma 4.2. Suppose that $\delta(G) = 2$. Then $1 + \frac{b}{2} \leq \Delta(G) \leq b$ by Lemma 4.3. If $\Delta(G) = b$, then $G \in \mathcal{G}_1(1, b) \subseteq \mathcal{G}_1(1, b)$ again by Lemma 4.3. If $1 + \frac{b}{2} \leq \Delta(G) < b$, then $b \neq 3$ and $\begin{cases} b = 4 \\ \Delta(G) = 3 \end{cases}$ or $\begin{cases} b = 5 \\ \Delta(G) = 4 \end{cases}$. Therefore, $G = S_{3,3}$ or $G \in \mathcal{G}_1(1, 3)$ if $b = 3$, that is $\mathcal{G}(1, 3) \subseteq \mathcal{G}_1(1, 3)$, and so $\mathcal{G}(1, 3) = \mathcal{G}_1(1, 3)$ (see Figure 5).

If $\begin{cases} b = 4 \\ \Delta(G) = 3 \end{cases}$, note that $\delta(G) = 2$, let $u, v \in V(G)$, $N(u) = \{x_1, x_2\}$ and $N(v) = \{y_1, y_2, y_3\}$. Applying (1) to u , we have $d(x_1) + d(x_2) = 6$, which implies that $(d(x_1), d(x_2)) = (3, 3)$. Applying (1) to v , we have $d(y_1) + d(y_2) + d(y_3) = 7$, which implies that $(d(y_1), d(y_2), d(y_3)) = (2, 2, 3)$. Let

$$\begin{aligned} C_1 &= \{u \in V(G) | d(u) = 3\}, \\ C_2 &= \{u \in V(G) | d(u) = 2\}. \end{aligned}$$

Then $\pi : V(G) = C_1 \cup C_2$ is a partition of $V(G)$. From the above arguments we know that $|N(u) \cap C_1| = 1$ and $|N(u) \cap C_2| = 2$ for any $u \in C_1$, $|N(u) \cap C_1| = 2$ and $|N(u) \cap C_2| = 0$ for any $u \in C_2$. Therefore, $\pi : V(G) = C_1 \cup C_2$ is a 2-equitable partition of $V(G)$ with parameters $(1, 2; 2, 0)$. By the definition, $G \in \mathcal{G}_2(1, 4) \subseteq \mathcal{G}_1(1, 4)$. Combining the arguments in the first paragraph, we know that $G = S_{4,4}$ or $G \in \mathcal{G}_1(1, 4)$ or $G \in \mathcal{G}_2(1, 4)$ if $b = 4$, that is $\mathcal{G}(1, 4) \subseteq \mathcal{G}_1(1, 4)$, and so $\mathcal{G}(1, 4) = \mathcal{G}_1(1, 4)$ (see Figure 6).

If $\begin{cases} b = 5 \\ \Delta(G) = 4 \end{cases}$, note that $\delta(G) = 2$, let $u, v \in V(G)$, $N(u) = \{x_1, x_2\}$ and $N(v) = \{y_1, y_2, y_3, y_4\}$. Applying (1) to u , we have $d(x_1) + d(x_2) = 7$. Since $\Delta(G) = 4$, we have $(d(x_1), d(x_2)) = (3, 4)$. Applying (1) to v , we have $d(y_1) + d(y_2) + d(y_3) + d(y_4) = 9$. Since $\delta(G) = 2$, we have $(d(y_1), d(y_2), d(y_3), d(y_4)) = (2, 2, 2, 3)$. Obviously, there is a vertex $w \in V(G)$ with $d(w) = 3$. Let $N(w) = \{z_1, z_2, z_3\}$. Applying (1) to w , we have $d(z_1) + d(z_2) + d(z_3) = 8$. Since $\delta(G) = 2$, we have $(d(z_1), d(z_2), d(z_3)) = (2, 2, 4)$ or $(2, 3, 3)$. Let

$$\begin{aligned} V_1 &= \{v \in V(G) | d(v) = 4, \text{ and the degrees of the neighbors of } v \text{ are } 2, 2, 2 \text{ and } 3\}, \\ V_2 &= \{v \in V(G) | d(v) = 3, \text{ and the degrees of the neighbors of } v \text{ are } 2, 2 \text{ and } 4\}, \\ V_3 &= \{v \in V(G) | d(v) = 3, \text{ and the degrees of the neighbors of } v \text{ are } 2, 3 \text{ and } 3\}, \\ V_4 &= \{v \in V(G) | d(v) = 2, \text{ and the degrees of the neighbors of } v \text{ are } 3 \text{ and } 4\}. \end{aligned} \tag{8}$$

Then $V_1 \cup V_2 \cup V_3 \cup V_4$ is a partition of $V(G)$. For convenience, if $|N(v) \cap V_j|$ is a constant for any $v \in V_i$, then set $r_{ij} = |N(v) \cap V_j|$. Note that if $r_{ij} = 0$ then $r_{ji} = 0$, and that $\sum_{j=1}^4 |N(v) \cap V_j| = d(v)$ for any $v \in V(G)$, combining (8) we have

$$\begin{cases} r_{11} = 0, & r_{12} = 1, & r_{13} = 0, & r_{14} = 3, \\ r_{21} = 1, & r_{22} = 0, & r_{23} = 0, & r_{24} = 2, \\ r_{31} = 0, & r_{32} = 0, & r_{33} = 2, & r_{34} = 1, \\ r_{41} = 1, & & & r_{44} = 0. \end{cases}$$

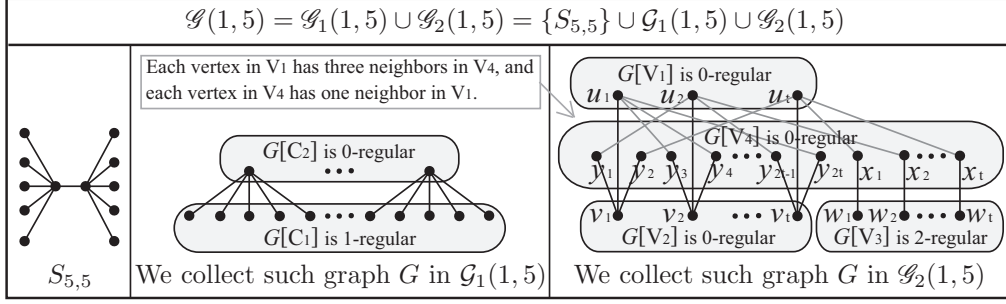


Figure 7: All the graphs in $\mathcal{G}(1, 5)$

Since $r_{ii} = 0 (i = 1, 2, 4)$, $G[V_i]$ is a vertex independent set for $i = 1, 2, 4$. Since $r_{33} = 2$, $G[V_3]$ is a 2-regular subgraph. Let $|V_1| = t$. Since $|V_1|r_{12} = |V_2|r_{21}$, we have $|V_2| = t$. Since $|V_1|r_{14} = |V_4|r_{41}$, we have $|V_4| = 3t$. Note that $|N(v) \cap V_2| + |N(v) \cap V_3| = 1$ for any $v \in V_4$, we have $|V_4| = |V_2|r_{24} + |V_3|r_{34}$, which gives $|V_3| = t$. Since $G[V_3]$ is a 2-regular subgraph, we have $t \geq 3$.

Let $V_1 = \{v_1^{(1)}, v_2^{(1)}, \dots, v_t^{(1)}\}$, $V_2 = \{v_1^{(2)}, v_2^{(2)}, \dots, v_t^{(2)}\}$, $V_3 = \{v_1^{(3)}, v_2^{(3)}, \dots, v_t^{(3)}\}$ and $V_4 = \cup_{i=1}^3 V'_{i4}$, where $V'_{i4} = \{u'_{i1}, u'_{i2}, \dots, u'_{it}\} (i = 1, 2, 3)$. Since $r_{12} = 1$ and $r_{21} = 1$, without loss of generality, let the edges of G between V_1 and V_2 are

$$\{v_i^{(1)}v_i^{(2)} \mid i = 1, \dots, t\}.$$

Since $r_{41} = 1$ and $r_{14} = 3$, without loss of generality, let the edges of G between V_1 and V_4 are

$$\{v_i^{(1)}u'_{ji} \mid i = 1, 2, \dots, t; j = 1, 2, 3\}.$$

Note that $r_{24} = 2$, $r_{34} = 1$ and $|N(v) \cap V_2| + |N(v) \cap V_3| = 1$ for any $v \in V_4$, we can relabeling the vertices in V_4 such that $V_4 = \cup_{i=1}^3 V_{i4}$, where $V_{i4} = \{u_{i1}, u_{i2}, \dots, u_{it}\} (i = 1, 2, 3)$, and the other edges of G are

$$\begin{cases} \text{the edges between } V_2 \text{ and } V_4: \{v_i^{(2)}u_{1i}, v_i^{(2)}u_{2i} \mid i = 1, \dots, t\}, \\ \text{the edges between } V_3 \text{ and } V_4: \{v_i^{(3)}u_{3i} \mid i = 1, \dots, t\}. \end{cases}$$

By the definition, $G \in \mathcal{G}_2(1, 5)$. Combining the arguments in the first paragraph, we know that $G = S_{5,5}$ or $G \in \mathcal{G}_1(1, 5)$ or $G \in \mathcal{G}_2(1, 5)$ if $b = 5$, that is $\mathcal{G}(1, 5) \subseteq \mathcal{G}_1(1, 5) \cup \mathcal{G}_2(1, 5)$, and so $\mathcal{G}(1, 5) = \mathcal{G}_1(1, 5) \cup \mathcal{G}_2(1, 5)$ (see Figure 7). \square

At the last of this section we propose a problem to characterize the graphs in $\mathcal{G}(a, b)$ for other feasible pairs (a, b) .

References

- [1] Cvetković D.M., The main part of the spectrum, divisors and switching of graphs, Publ. Inst. Math. (Beograd) (N.S.), 23(37), 31–38, (1978).
- [2] Rowlinson P., The main eigenvalues of a graph: a survey, Appl. Analysis and Discete Math, 1, 445–471, (2007).
- [3] Hagos E.M., Some results on graph spectra, Linear Algebra Appl., 356, 103–111, (2002).
- [4] Hou Y.P., Tian F., Unicyclic graphs with exactly two main eigenvalues, Appl. Math. Lett., 19, 1143–1147, (2006).
- [5] Teranishi Y., Main eigenvalues of a graph, Linear and Multilinear Algebra, 49(4), 289–303, (2001).
- [6] Hou Y.P., Zhou H.Q., Trees with exactly two main eigenvalues, Acta of Hunan Normal University, 28(2), 1–3, (2005) (in Chinese).

- [7] Shi L.S., On graphs with given main eigenvalues, *Appl. Math. Lett.*, 22, 1870–1874, (2009).
- [8] Hu Z.Q., Li S.C., Zhu C.F., Bicyclic graphs with exactly two main eigenvalues, *Linear Algebra Appl.*, 431, 1848–1857, (2009).
- [9] Hou Y.P., Tang Z.K., Shiu W.C., Some results on graphs with exactly two main eigenvalues, *Appl. Math. Lett.*, 25, 1274–1278, (2012).
- [10] Tang Z.K., Hou Y.P., Tricyclic Graphs with Exactly Two Main Eigenvalues, *Acta of Hunan Normal University*, 34(4), 7–12, (2011) (in chinese).
- [11] Fan X.X., Luo Y.F., Gao X., Tricyclic graphs with exactly two main eigenvalues, *Central European Journal of Mathematics*, 11(10), 1800–1816, (2013).
- [12] Tang Z.K., Hou Y.P., The integral graphs with index 3 and exactly two main eigenvalues, *Linear Algebra Appl.*, 433, 984–993, (2010).